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A Flexible Parametrization of CKM matrix via Singular-Value-Decomposition Method

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Abstract

We investigate a flexible method in which we can test the unitarity of the quark flavor mixing matrix step-by-step. Singular-Value-Decomposition (SVD) techniques are used in analyzing the mixing matrix over a broader parameter region than the unitarity region. Unitary constraints make us extract CP violating properties without any specific parametrization when the magnitudes of at least three mixing matrix elements in three generation quark mixing are given. This method can also be applied to the analysis of lepton flavor mixing, in which only a few moduli are presently measured.

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I. INTRODUCTION

The Cabibbo-Kobayashi-Maskawa (CKM) [1,2] matrix makes us possible to explain all flavor changing weak decay processes and CP violating phenomena up to now. Unitarity of the CKM matrix in the standard model (SM) is a unique property that we cannot loosen. We can use any parametrization of the CKM matrix as long as its unitarity is conserved. The original parametrization for three generation quark mixing is the Kobayash-Maskawa (KM) parametrization. The standard parametrization proposed by Chau and Keung [3,4] is the product of three complex rotation matrices which are characterized by the three Euler angles, $\theta_{12},\ \theta_{13},\ \theta_{23}$ and an CP–violating phase δ_{13}. More widely used one is the Wolfenstein parametrization [5], which was suggested as a simple expansion of the CKM matrix in terms of the four parameters: $\lambda,\ A,\ \rho$ and η. It has been also known that the CKM matrix for the three-generation case can be parameterized in terms of the moduli of four of its elements [6]. This four-value-KM (4VKM) parametrization is rephasing invariant and directly related to the measured quantities. In three generation case we always need four independent parameters to define a unitary 3×3 matrix, as explained, eg. $\theta_{12},\ \theta_{13},\ \theta_{23}$ and δ_{13}, or $\lambda,\ A,\ \rho$ and η or even only moduli of any four independent elements of the matrix.

The 4VKM parametrization has several advantages over the other parametrization. This parametrization doesn't need any specific representations for the mixing angles as long as the CKM is unitary, and no ambiguity over the definition of its complex CP phase is present above all. Secondly, the Jarlskog invariant quantity J_{cp} and non-trivial higher invariants can be reformulated as functions of moduli and quadri-products [7]. However, in the 4VKM parametrization initial four-moduli input values should be fixed by experiments. Once we set four moduli to specific values, remaining five moduli of mixing elements are automatically fixed and we may lose some characteristic effects from interplaying between the moduli. In a conceptual point of view it is better if we can reduce number of *a priori* experimental input

values. This paper presents a novel parametrization in which we start with three-moduli input values. Through simple algebraic relations we can determine remaining six moduli of mixing elements. With more broader parameter space we can check the compatibility between measured values of mixing elements and their unitarity properties step-by-step.

Many groups have made global fits and numerical works on CKM matrix elements with conventional representations which satisfy unitarity [8]. One of the problems in these conventional parameterizations is that they are *fully* and *completely* unitary and are not flexible to include possible non-unitary properties resulted from unknown new physics. Therefore, it is a complicate task to make a step-by-step test to check the unitarity with experimental data if you use a unitary parametrization. In the following, we present three extended definitions for the unitarities of mixing matrix V in the order of the strength of the constraints:

- Weak Unitary Conditions (WUC): We define that the mixing matrix V is weak unitary if it satisfies

$$\sum_{\alpha} |V_{i\alpha}|^2 = \sum_{j} |V_{j\beta}|^2 = 1 \text{ for all } i = u, c, t, \text{ and } \quad \beta = d, s, b. \tag{1}$$

  These constraints appear to be well satisfied experimentally for the three generation case, and we start from this. Actually it was pointed out that there is an apparent functional violation in the available data: $|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 < 1$ [9]. For such a case with $\sum_{\alpha} |V_{u\alpha}|^2 = a < 1$, we can easily generalize our method, and we just start with this new condition.

- Almost Unitary Conditions (AUC): In addition to the constraint Eq. (1), if the following constraints are satisfied

$$\sum_{\alpha, i \neq j} V_{i\alpha}^* V_{j\alpha} = \sum_{j, \alpha \neq \beta} V_{j\alpha}^* V_{j\beta} = 0 \text{ for some parts of } i, j = u, c, t, \text{ and } \quad \alpha, \beta = d, s, b, \tag{2}$$

  let us call the mixing matrix almost unitary. Some combinations, which do not satisfy Eq. (2), may not make closed triangles, and may have different areas even though

making closed triangles. We have no specific models in which the mixing matrix satisfies this almost unitary conditions. Therefore, we will not consider the case with AUC.

- Full Unitary Conditions (FUC): This corresponds to usual unitarity in which Eqs. (1), (2) are satisfied *for all the indices*. All six unitarity triangles from Eq. (2) have the same areas.

In Sec. II, we propose an alternative and more flexible parametrization of the CKM matrix in terms of the three moduli and the one independent parameter, which is induced by the singular-value-decomposition (SVD) method. We describe how to get the new parametrization of the CKM matrix by using the SVD method in the three-generation case. Unlike from the previous parametrization with four moduli [6], we have more flexible leverage to test the unitarity step-by-step. We start with only three moduli rather than four moduli, and the remaining one can be adjusted depending on the condition of the unitarity, which we apply, *i.e.* WUC or FUC. In Sec. III, we analyze the CKM matrix numerically with our parametrization with the SVD method. Conclusions are also in Sec. III. Appendices A–B include details about the SVD method.

II. NEW PARAMETRIZATION OF THE CKM MATRIX BY THE SVD METHOD

We start with a definition in such a way that it satisfies the weak unitary conditions, Eq. (1): we have six constraint equations for the 3 generation mixing. These constraints are only parts of unitarity conditions and the introduced mixing matrix V may not be fully unitary. We study this explicitly with three generation quark flavor mixing matrix V in their absolute values and choose three independent moduli as starting input parameters. Explicit analysis depends on the choice of three input parameters. We consider the case with:

- Our Choice (Set–A): Input parameters $|V_{us}|,\ |V_{ub}|,\ |V_{cb}|$.

We can also choose different sets of input parameters, as examples:

- Set–B: Input parameters $|V_{ud}|,\ |V_{us}|,\ |V_{cd}|$,
- Set–C: Input parameters $|V_{us}|,\ |V_{cs}|,\ |V_{cb}|$.

Mathematically three parametrizations of Set–A,B,C are all equivalent if three input values of each set are independent one another and all equally precisely measured. However, in reality, the upper-left 2×2 part of CKM matrix is *approximately* unitary and only one independent variable is dominantly evident, for example, the parameter λ in the Wolfenstein parametrization or the Cabibbo angle $\sin\theta_c$. Therefore, Set–B would be the worst choice for numerical analyses. For our choice of Set–A, the three inputs are all off-diagonal and independent each other, and all three values can be determined by three semileptonic decays, in which new physics contributions are severely suppressed. Therefore, we select upper off-diagonal elements in V, namely, $|V_{us}|, |V_{ub}|, |V_{cb}|$ as the initial input variables in our analysis, *i.e.* the case with Set–A.

If we are given the three input values of Set–A, then we get the values $|V_{ud}|$ and $|V_{tb}|$:

$$|V_{ud}|^2 = 1 - |V_{us}|^2 - |V_{ub}|^2, \tag{3}$$

$$|V_{tb}|^2 = 1 - |V_{ub}|^2 - |V_{cb}|^2. \tag{4}$$

To obtain four remaining elements,$|V_{cd}|, |V_{cs}|, |V_{td}|$ and $|V_{ts}|$, we write four constraints for these four elements in Eq. (1) as a matrix form:

$$RX = B, \tag{5}$$

where

$$R = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \tag{6}$$

$$X = (|V_{cd}|^2, |V_{cs}|^2, |V_{td}|^2, |V_{ts}|^2)^T, \tag{7}$$

$$B = (1 - |V_{cb}|^2, 1 - |V_{tb}|^2, 1 - |V_{ud}|^2, 1 - |V_{us}|^2)^T. \tag{8}$$

In Eqs. (7), (8), X and B are column vectors and T means transpose of the matrix. Because of det$R$=0, there is not a unique solution if any. In such a situation there exists a very powerful set of technique, known as Singular Value Decomposition (SVD) method. The details of the method are given in the Appendix A. Remaining mixing elements are expressed as follows:

$$\begin{aligned} |V_{cd}|^2 &= -a + u_1, \\ |V_{cs}|^2 &= a + u_2, \\ |V_{td}|^2 &= a + u_3, \\ |V_{ts}|^2 &= -a + u_4, \end{aligned} \tag{9}$$

where

$$\begin{aligned} u_1 &= \frac{1}{4}(1 + 2|V_{us}|^2 + |V_{ub}|^2 - 2|V_{cb}|^2), \\ u_2 &= \frac{1}{4}(3 - 2|V_{us}|^2 - |V_{ub}|^2 - 2|V_{cb}|^2), \\ u_3 &= \frac{1}{4}(-1 + 2|V_{us}|^2 + 3|V_{ub}|^2 + 2|V_{cb}|^2), \\ u_4 &= \frac{1}{4}(1 - 2|V_{us}|^2 + |V_{ub}|^2 + 2|V_{cb}|^2), \end{aligned} \tag{10}$$

and a new variable, 'a', is introduced as a coefficient attached to the general solution. If there is no flavor mixing, we can set $a = 1/4$. The value of 'a' can be determined from Eq. (9) if we know any one value of $|V_{cd}|, |V_{cs}|, |V_{td}|, |V_{ts}|$. Constraints of non-negative $|V_{ij}|^2$

are applied for the range of variable a:

$$a_{min} = \max(-u_2, -u_3), \quad a_{max} = \min(u_1, u_4).$$

We note that when three input values $|V_{us}|, |V_{ub}|, |V_{cb}|$ are given, the moduli squared of remaining four mixing elements $|V_{cd}|, |V_{cs}|, |V_{td}|, |V_{ts}|$ are just quadratic functions of parameter a. As 'a' increases, $|V_{cs}|$ and $|V_{td}|$ increase while $|V_{cd}|$ and $|V_{ts}|$ decrease. $|V_{ud}|$ and $|V_{tb}|$ are fixed by the three input values and are independent from the parameter a. And the bounds on the parameter 'a' will determine the regions of FUC and WUC, which will be explained later.

As a next step, we further assume that the mixing matrix V satisfies full unitary conditions. Then we have six more constraints:

$$\begin{aligned} \sum_{j=d,s,b} V_{ij} V_{kj}^* &= 0, \qquad (i,k) = (u,c), (u,t), (c,t), \\ \sum_{j=u,c,t} V_{ji} V_{jk}^* &= 0, \qquad (i,k) = (d,s), (d,b), (s,b). \end{aligned} \tag{11}$$

These constraints cannot be represented without introduction of complex numbers analytically. If we know all the absolute values of V, however, we can express necessary and sufficient conditions for the constraints, Eqs. (11), in a geometric way. Eqs. (11) give six unitarity triangles corresponding to each six constraints, and all six triangles have equal area that is directly related to the Jarlskog's rephasing invariant parameter J_{CP}. If we take one of the constraints Eqs. (11), for example,

$$\sum_{j=u,c,t} V_{jd} V_{jb}^* = 0,$$

a triangle is composed of three sides with lengths $|V_{ud}||V_{ub}|, |V_{cd}||V_{cb}|$, and $|V_{td}||V_{tb}|$, with a necessary condition

$$|V_{cd}||V_{cb}| \leq |V_{ud}||V_{ub}| + |V_{td}||V_{tb}|, \tag{12}$$

where the equality holds in CP conserving case. For more general argument, let us rewrite Eq. (12) as follows:

$$l_2 \leq l_1 + l_3, \tag{13}$$

where, as an example, $l_1 = |V_{ud}||V_{ub}|, l_2 = |V_{cd}||V_{cb}|$, and $l_3 = |V_{td}||V_{tb}|$. After taking the square on both sides of Eq. (13) we can rearrange the constraint equation as follows:

$$f(l_1, l_2, l_3) \equiv 2l_1^2 l_2^2 + 2l_2^2 l_3^2 + 2l_1^2 l_3^2 - l_1^4 - l_2^4 - l_3^4 \geq 0, \tag{14}$$

where we denote newly introduced function f for later use. Using the Heron's formula, the square of triangular area can be rewritten as follows:

$$A^2 = s(s - l_1)(s - l_2)(s - l_3) = \frac{1}{16} f(l_1, l_2, l_3), \tag{15}$$

where $s = (l_1 + l_2 + l_3)/2$. So the necessary condition (14) for the complete triangle means non-negative value of A^2. The Jarlskog's invariant parameter is written as follows:

$$J_{CP} = 2A = \frac{1}{2}\sqrt{f(l_1, l_2, l_3)}. \tag{16}$$

If we expand f in terms of parameter a:

$$\begin{aligned} f = & -(1 - |V_{ub}|^2)^2 a^2 \\ & +2[|V_{ud}|^2 |V_{ub}|^2 (|V_{tb}|^2 - |V_{cb}|^2)(|V_{cb}|^2 u_1 - |V_{tb}|^2 u_3)(|V_{cb}|^2 + |V_{tb}|^2)]a \\ & +2|V_{ud}|^2 |V_{ub}|^2 [|V_{cb}|^2 u_1 + |V_{tb}|^2 u_3] - (|V_{cb}|^2 u_1 - |V_{tb}|^2 u_3)^2 - |V_{ud}|^4 |V_{ub}|^4, \end{aligned} \tag{17}$$

where the function f is quadratic of a. We can get the boundaries of the constraint Eq. (13), and denote two roots of the quadratic equation as a_- and $a_+ (> a_-)$. Two real roots and the boundary points in the interval depend on only the three input values. Nonexistence of real solutions of the quadratic equation means that the three input values do not allow the

FUC. We note that if we force the mixing matrix V to be fully unitary, then six triangles from the constraints Eq. (11) have the same area, which are the sufficient conditions for the FUC.

We can relate the coefficient a to the CP violating parameter in another representation of mixing matrix with the FUC. Let us consider the standard parametrization of the CKM matrix

$$V_{CKM} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix}, \tag{18}$$

where $s_{ij} = \sin\theta_{ij}, c_{ij} = \cos\theta_{ij}$. We find that the coefficient a is directly related to the parameters in the standard representation:

$$a = -2s_{12}c_{12}s_{23}c_{23}s_{13}\cos\delta_{13} - \frac{1}{8}\cos 2\theta_{12}\cos 2\theta_{23}[-3 + \cos 2\theta_{13}]. \tag{19}$$

Three angles α, β, γ of the unitarity triangle, which characterize CP violation, are defined as follows:

$$\alpha = Arg[-(V_{td}V_{tb}^*)/(V_{ud}V_{ub}^*)], \tag{20}$$

$$\beta = Arg[-(V_{cd}V_{cb}^*)/(V_{td}V_{tb}^*)], \tag{21}$$

$$\gamma = Arg[-(V_{ud}V_{ub}^*)/(V_{cd}V_{cb}^*)]. \tag{22}$$

The sum of those three angles, defined as the intersections of three lines, would be always equal to 180^0, even though the three lines may not be closed to make a triangle, *i.e.* in case that CKM matrix is not unitary at all. We can also define these quantities from the area of the unitary triangle and its sides:

$$\sin\beta' = \frac{2A}{|V_{td}||V_{tb}||V_{cd}||V_{cb}|}, \tag{23}$$

$$\sin\gamma' = \frac{2A}{|V_{ud}||V_{ub}||V_{cd}||V_{cb}|}, \tag{24}$$

$$\alpha' = \pi - \beta' - \gamma', \tag{25}$$

when the FUC is fully satisfied and the area of the triangles can be defined from (15). Any experimental data that indicates $\alpha \neq \alpha'$ or $\beta \neq \beta'$ or $\gamma \neq \gamma'$ means that three generation quark mixing matrix V is not fully unitary.

III. NUMERICAL RESULTS AND DISCUSSIONS

For given input values of $|V_{us}|, |V_{ub}|, |V_{cb}|$, the parameter '$a$' is divided into two regions depending on whether the FUC is satisfied or not. We can divide the range of the parameter into two by setting $l_1 = |V_{ud}||V_{ub}|, l_2 = |V_{cd}||V_{cb}|, l_3 = |V_{td}||V_{tb}|$, as an example:

- Region I : The maximum among l_1, l_2, l_3 is larger than sum of the other two values. In other words it is not possible to make any triangle with these three segments. This region is outside of the interval of (a_-, a_+).

- Region II : The maximum among l_1, l_2, l_3 is smaller than sum of the other two values. In other words it is possible to make a unitarity triangle. This region is confined to (a_-, a_+).

In region I, we cannot define J_{CP}. On the contrary we can define J_{CP} in region II and calculate it with l_1, l_2, l_3 as shown in Eq. (15). In general the region II is surrounded by the region I. Two boundary points of region II correspond to the case of CP conserving case.

For numerical analyses, we refer to the Particle Data Group (PDG) [4]. Current values of three input moduli and corresponding sources of measured matrix elements are summarized in Table I. The input values of $|V_{us}|, |V_{ub}|, |V_{cb}|$ are randomly generated within 95% CL with

TABLE I. Input values of the matrix elements and their sources referred from the PDG. The output values are allowed intervals (95% CL) for WUC and FUC.

	matrix elements	PDG values	Sources	
Input	$\|V_{us}\|$	0.2196 ± 0.0026	K_{e3} decays	
	$\|V_{ub}\|$	$(3.6 \pm 0.7) \times 10^{-3}$	B semileptonic decays	
	$\|V_{cb}\|$	$(41.2 \pm 2.0) \times 10^{-3}$	B semileptonic decays	
	matrix elements	WUC	FUC	PDG
Output	$\|V_{cd}\|$	$0.210 \sim 0.224$	$0.214 \sim 0.224$	$0.219 \sim 0.226$
	$\|V_{cs}\|$	$0.9735 \sim 0.9768$	$0.9735 \sim 0.9760$	$0.9732 \sim 0.9748$
	$\|V_{td}\|$	$0.004 \sim 0.045$	$0.004 \sim 0.014$	$0.004 \sim 0.014$
	$\|V_{ts}\|$	$0.001 \sim 0.045$	$0.035 \sim 0.045$	$0.037 \sim 0.044$

uniform distributions. Each input determines both two regions for the WUC and FUC. The WUC is confined to the interval (a_{min}, a_{max}) of which calculations are described in the previous section. The FUC is confined into the interval (a_-, a_+) which is a subset of (a_{min}, a_{max}). In the restricted regions we again generated randomly the values of parameter a for our numeric calculations. Fig. 1(a) shows scattered points for $|V_{td}|$ and $|V_{ts}|$ values when we apply the WUC to the choice of parameter a. The scattered points compose a quadrant in the $|V_{td}|$–$|V_{ts}|$ plane. In the figure we also draw the curved axis for the parameter a. The labels on the curve are valid only when we set the three inputs to the center values in Table I. If other input values are taken, the numeric labelling should be slightly changed. Fig. 1(b) presents scattered points when we apply the FUC to the choice of parameter a. The allowed region for FUC is much narrower than that for the WUC and is included in the region for the WUC. The curved axis for the parameter a is identical to that in Fig. 1(a). Fig. 2 shows corresponding scattered points for $|V_{ud}|$ and $|V_{cd}|$ when we take the WUC and FUC. The moduli $|V_{ud}|$ does not depend on the parameter a and the $|V_{cd}|$ is directly related to the value of a. The axis for the parameter a is, therefore, a vertical line along the axis of